Math 10A with Professor Stankova
Midterm II Review; **Monday, 10/30/2017**
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Riemann Sums

1. **TRUE** False Definite integrals are shorthand for Riemann sums.

> **Solution:** Definite integrals are just a way for us to succinctly write a limit of Riemann sums.

2. **TRUE** False The jumping constant definite integral law comes from the corresponding limit law.

> **Solution:** Since definite integrals are just limits of Riemann sums, the laws are derived from limit laws.

3. True **FALSE** A limit of Riemann sums depends on what method we use (Left endpoint, etc.).

4. **TRUE** False The second derivative can tell us if the midpoint rule gives an over/under estimate.

> **Solution:** If the second derivative is always positive, then the midpoint rule gives an overestimate, and if the second derivative is always negative, the midpoint rule gives an underestimate.

5. Express $\lim_{n\to\infty}[\frac{2^2}{n^2} + \frac{2\cdot 2^2}{n^2} + \cdots + \frac{2^2 n}{n^2}]$ as a definite integral from 0 to 2.

> **Solution:** If we divide up the interval of $[0,2]$ into n intervals, then each subinterval is of length is $\frac{2-0}{n} = \frac{2}{n}$. Using right endpoints, the first term is $(\frac{2}{n}) \cdot \frac{2}{n}$ and hence we see that $f(x) = x$. Thus, this limit is
>
> $$\int_0^2 x dx.$$

6. Express $\int_0^3 \cos^2(x)dx$ as a limit of right endpoint Riemann sums.

Solution: We split up the interval $[0,3]$ into n intervals which are each of length $\frac{3-0}{n} = \frac{3}{n}$. We start at 0 and end at 3 to get $[0,3/n], [3/n,6/n], \dots, [3-3/n,3]$. Using the right endpoint method, we have that the integral is the limit

$$\lim_{n\to\infty}\left[\cos^2(3/n)\cdot\frac{3}{n}+\cos(6/n)\cdot\frac{3}{n}+\cdots+\cos(3)\cdot\frac{3}{n}\right] = \lim_{n\to\infty}\sum_{i=1}^{n}\cos^2(\frac{3i}{n})\cdot\frac{3}{n}.$$

FTC

7. **TRUE** False The crown of calculus is the fundamental theorem of calculus.
8. **TRUE** False FTC I gives us an easy way to explicitly calculate definite integrals.

Solution: FTC I tells us that if F is an antiderivative of f, then $\int_a^b f(x)dx = F(b) - F(a)$, so we don't need to calculate a limit of Riemann sums.

9. **TRUE** False $\int_a^x e^{t^2}dt$ is an antiderivative of e^{x^2}.

Solution: This is true by FTC II. The derivative of $\int_a^x e^{t^2}dt$ is e^{x^2} by FTC II. But, there is no way to write it in terms of elementary functions.

10. True **FALSE** FTC I and FTC II are not related.

Solution: Let $F(x)$ be an antiderivative of $f(x)$. By FTC I, we have that $\int_a^x f(t)dt = F(x) - F(a)$ and hence $\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}(F(x)-F(a)) = F'(x) = f(x)$, which "proves" FTC II.

11. Find an antiderivative of $e^{e^{x^2}}$ with $F(0) = 1$ (not necessarily in elementary functions).

Solution:

$$F(x) = \int_0^x e^{e^{t^2}} dt + 1$$

satisfies the condition because $F(0) = 0 + 1$.

U-Substitution/Integration by Parts

12. True **FALSE** When integrating by parts, choosing different functions for u and dv (assuming both work out), will give different answers.

13. **TRUE** False It is always good to u sub first in order to simplify the integral.

Solution: This is true and u subbing first will make your life a lot easier.

14. Find $\int_0^1 \sqrt{1-\sqrt{x}}dx$.

Solution: We guess that $u = \sqrt{1-\sqrt{x}}$ and hence $u^2 = 1 - \sqrt{x}$ so $\sqrt{x} = 1 - u^2$ and $x = (1-u^2)^2 = u^4 - 2u^2 + 1$. Thus, we have that $dx = (4u^3 - 4u)du$. When $x = 0$, then $u = \sqrt{1-0} = 1$ and when $x = 0$, then $u = \sqrt{1-\sqrt{1}} = 0$. Thus

$$\int_0^1 \sqrt{1-\sqrt{x}}dx = \int_2^0 u(4u^3 - 4u)du = \frac{4}{5}u^5 - \frac{4}{3}u^3\Big|_1^0 = (0-0) - (4/5 \cdot 1^5 - 4/3 \cdot 1^3)$$

$$= \frac{4}{3} - \frac{4}{5}.$$

15. Find $\int x^5 e^{x^3} dx$.

Solution: We guess that $u = x^3$ and so $du = 3x^2$. So $x^5 e^{x^3} = x^3/3e^u du = u/3e^u du$ so

$$\int x^5 e^{x^3} dx = \int \frac{ue^u}{3} du.$$

We use integration by parts to get that this is equal to

$$\frac{ue^u - e^u + C}{3} = \frac{x^3 e^{x^3} - e^{x^3}}{3} + C.$$

Numerical Integration

16. True **FALSE** Numerical approximations are just approximations, and never the exact answer.

Solution: Any approximation for a constant will give the exact answer.

17. **TRUE** False Simpson's method will approximate cubics exactly.

Solution: The error bound is given by K_4, which is the maximum of the fourth derivative. Since the fourth derivative of cubics is 0, the error is 0.

18. True **FALSE** When calculating K_1 of $f(x)$ on $[a, b]$, we have that K_1 is the maximum of $|f'(a)|$ and $|f'(b)|$.

Solution: K_1 is the maximum of $|f'(x)|$ on the interval $[a, b]$, which may not occur at the endpoints.

19. How many intervals do we need to use to approximate $\int_1^4 \ln x dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We have that $E_S = \frac{K_4(b-a)^5}{180n^4}$ and $(\ln x)^{(4)} = \frac{-6}{x^4}$ so $10^{-3} = \frac{6(3)^5}{180n^4}$ so $n \geq 9.49$. So the minimum number is $n = 10$.

Differential Equations

20. **TRUE** False We can determine the behavior of solutions of a differential equation without explicitly solving for them.

Solution: We can by finding steady states and going from there.

21. The rate at which an animal loses heat is proportional to its surface area (L^2). If the amount of heat an animal has is proportional to its volume (L^3), write this as a differential equation in terms of heat H.

Solution: We know that $\frac{dH}{dt} = -kL^2$ and $H = CL^3$. So solving for L gives us $L = \sqrt[3]{H/C}$ and $L^2 = D\sqrt[3]{H^2}$, for another constant D. Combining this with the previous equation gives $\frac{dH}{dt} = -kD\sqrt[3]{H^2} = -K\sqrt[3]{H^2}$ for some constant K.

22. The rate at which a person grows is proportional to his current height multiplied by his maximum height L minus his current height. Write this as a differential equation.

Solution:

$$\frac{dH}{dt} = kH(L - H).$$

Separable Equations

23. True **FALSE** When solving a separable equation, if we get that $ydy = xdx$, then the solution is $y = x + C$.

Solution: Solving gives $y^2/2 = x^2/2 + C$ and multiplying by two and square rooting gives $y = \sqrt{x^2 + 2C}$, which is not the same as $y = x + C$.

24. **TRUE** False When solving a separable equation, we need to put the $+C$ immediately after integration.

Solution: This is true. We cannot wait to place the $+C$ at the very end because this will give the wrong solution.

25. Find the general solution to $\frac{dy}{dx} = e^{x-y}$.

Solution: We have that $e^{x-y} = e^x/e^y$ and hence

$$e^y dy = e^x dx \implies e^y = e^x + C \implies y = \ln(e^x + C).$$

Note that this is not the same as $y = x + C$.

Improper Integrals

26. True **FALSE** We can compare an integral to $\int_1^\infty 1/\sqrt{x}dx$ in order to show it converges.

Solution: The given integral diverges and hence cannot be used to show an integral converges.

27. True **FALSE** We can compare an integral to $\int_1^\infty 1/x^2 dx$ to show it diverges.

Solution: The given integral converges and hence cannot be used to show that another integral diverges.

28. True **FALSE** Since $x < x + 1$, we have that $\infty = \int_1^\infty \frac{1}{x}dx \leq \int_1^\infty \frac{1}{x+1}dx$ so the latter integral diverges.

Solution: When we take reciprocals, we need to switch the sign so we actually get $\infty \int_1^\infty \frac{1}{x}dx \geq \int_1^\infty \frac{1}{x+1}dx$ so we don't have any information on if the latter integral converges or not. It does in fact diverge but we need to show that a different way.

29. Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+(x-1)^3}dx$.

Solution: We have to split up the integral first but it doesn't matter where we do so. We choose $x = 1$ for simplicity.

$$\int_{-\infty}^{\infty} \frac{1}{1+(x-1)^2}dx = \int_{-\infty}^{1} \frac{1}{1+(x-1)^2}dx + \int_{1}^{\infty} \frac{1}{1+(x-1)^2}dx$$

$$= \lim_{t\to-\infty} \int_t^1 \frac{1}{1+(x-1)^2}dx + \lim_{r\to\infty} \int_1^r \frac{1}{1+(x-1)^2}dx = \lim_{t\to-\infty} \arctan(x-1)|_t^1 + \lim_{r\to\infty} \arctan(x-1)|_1^r$$

$$= \arctan(0) - \arctan(-\infty) + \arctan(\infty) - \arctan(0) = \pi/2 - (-\pi/2) = \pi.$$

Histograms

30. True **FALSE** The height of each bar of a histogram represents the percentage of people that fall under that bin.

Solution: The area of each bar represents the percentage, not the height.

31. **TRUE** False The bars of a histogram can have a height greater than 1.

Solution: If all of the data falls within an interval of 0.1, then the height of that bar is $1/0.1 = 10$.

PDFs/CDFs

See Worksheet 28.